Supersymmetric Two-Boson Equation: Bilinearization and Solutions

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Abstract

A bilinear formulation for the supersymmetric two-boson equation is derived. As applications, some solutions are calculated for it. We also construct a bilinear Bäcklund transformation.

1 Introduction

The whole theory of solitons originated from a single partial differential equation – the celebrated Korteweg-de Vries (KdV) equation, which is regarded as one of the most important systems in mathematics and physics. A supersymmetric extension of the KdV equation was introduced by Manin and Radul [16] (see also [18]). Much research has been conducted on this $N = 1$ supersymmetric KdV system and many remarkable properties have been established. Here we mention just a few, bi-Hamiltonian structures[22], Painlevé property[19], infinite many symmetries, Darboux transformation [14] and Bäcklund transformation (BT) [15] and bilinear forms [20][5][6].

Another equally important or more remarkable system is the Broer-Kaup (BK) system [4][10], which was solved by the inverse scattering transformation and certain soliton solutions are found. We remark here that both the classical Boussinesq (cB) system and the dispersive water wave (DWW) equation are equivalent to the BK equation. Kupershmidt in [12] constructed three local Hamiltonian structures for the BK or DWW system and its Bäcklund and Darboux transformations are studied in [13]. This system has various interesting solutions, such as standard solitons [10][9], fusion and fission solitions [17][23] and rational solutions [21]. We also mention that the two-boson equation is one more name for this system. It is shown that this equation appears in matrix model theory [1][3].

A supersymmetric version of this system is proposed by Brunelli and Das [2] and reads

as

$$\begin{cases} \Phi_{0,t} = -(\mathcal{D}^4\Phi_0) + (\mathcal{D}(\mathcal{D}\Phi_0)^2) + 2(\mathcal{D}^2\Phi_1), \\ \Phi_{1,t} = (\mathcal{D}^4\Phi_1) + 2(\mathcal{D}^2((\mathcal{D}\Phi_0)\Phi_1)), \end{cases} \tag{1}$$

where Φ_0 and Φ_1 are fermionic superfields depending on usual independent variables x and t and the Grassmann variable θ. $\mathcal{D} = \partial_\theta + \theta\partial$ is the usual super derivative. The system is shown to be a bi-Hamiltonian system, have a Lax representation and various reductions [2].

The purpose of the present Note is to study the supersymmetric two boson (sTB) from the viewpoint of solutions. We will show that the system can be casted into Hirota's bilinear form and this in turn provides us a way to find solutions.

The Note is organized as follows. In the next section, we will transform the sTB equation into bilinear form. Then section 3 will be devoted the construction of solutions. In section 4, we construct a bilinear Bäckulund transformation for the sTB system. Final section contains our discussion and conclusion.

2 Bilinear Form

To obtain a bilinearization of the sTB system (1), we first reformulate it. Let

$$u = \mathcal{D}\Phi_0, \quad \alpha = \Phi_1$$

then the system (1) is transformed into

$$\begin{cases} u_t = -u_{xx} + 2uu_x + 2(\mathcal{D}\alpha_x), \\ \alpha_t = \alpha_{xx} + 2(u\alpha)_x. \end{cases} \tag{2}$$

Now we introduce the following transformations for the dependent variables

$$u = \left(\ln\frac{\tau_2}{\tau_1}\right)_x, \quad \alpha = (\mathcal{D}\ln\tau_2)_x, \tag{3}$$

substituting above expressions into the first equation of (2), we obtain

$$\left[\frac{1}{\tau_1\tau_2}(\tau_{1,t}\tau_2 - \tau_1\tau_{2,t} + \tau_{1,xx}\tau_2 - 2\tau_{1,x}\tau_{2,x} + \tau_1\tau_{2,xx})\right]_x = 0$$

which gives us

$$(D_t + D_x^2)\tau_1 \cdot \tau_2 = 0 \tag{4}$$

where the Hirota derivative is defined as

$$D_t^m D_x^n f \cdot g = \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1, t_1, \theta_1) g(x_2, t_2, \theta_2) \Big|_{\substack{x_1=x_2 \\ t_1=t_2}} .$$

Let us now turn to the second equation of (2). After substituting (3) into it and some calculations, we obtain

$$\begin{aligned}\Bigg\{ &\frac{(\mathcal{D}\tau_2)}{\tau_1\tau_2^2}\left[\tau_1\tau_{2,t} - \tau_1\tau_{2,xx} + 2\tau_{1,x}\tau_{2,x}\right] \\ &\qquad - \frac{1}{\tau_1\tau_2}\left[\tau_1(\mathcal{D}\tau_{2,t}) - \tau_1(\mathcal{D}\tau_{2,xx}) + 2\tau_{1,x}(\mathcal{D}\tau_{2,x})\right]\Bigg\}_x = 0. \end{aligned} \tag{5}$$

It is interesting that the left hand side of the equation (5) can be represented as

$$\left\{\frac{1}{2\tau_1\tau_2}\left[\mathcal{D}((D_t + D_x^2)\tau_1\cdot\tau_2) - (SD_t + SD_x^2)(\tau_1\cdot\tau_2)\right] - \frac{(\mathcal{D}\tau_2)}{\tau_1\tau_2^2}\left[(D_t + D_x^2)\tau_1\cdot\tau_2\right]\right\}_x$$

therefore, taking account of the equation (4) we obatin the other bilinear equation

$$(SD_t + SD_x^2)\tau_1\cdot\tau_2 = 0 \tag{6}$$

where we used a super version of Hirota derivative which is introduced in [20] [5]. Its definition is following

$$\begin{aligned} &SD_t^m D_x^n f\cdot g = \\ &\quad (\mathcal{D}_{\theta_1} - \mathcal{D}_{\theta_2})\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1, t_1, \theta_1)g(x_2, t_2, \theta_2)\Big|_{\substack{x_1 = x_2\\ t_1 = t_2\\ \theta_1 = \theta_2}}. \end{aligned}$$

Thus we have succeeded in converting the sTB system (1) into Hirota's bilinear form. For convenience, we collect them below

$$\begin{cases} (D_t + D_x^2)\tau_1\cdot\tau_2 = 0, \\ (SD_t + SD_x^2)\tau_1\cdot\tau_2 = 0. \end{cases} \tag{7}$$

3 Solutions

It is well known that Hirota's bilinear form is ideal for constructing interesting particular solutions. Next we shall show that a class of solutions can be calculated for the sTB equation. To this end, let

$$\tau_1 = \epsilon f, \quad \tau_2 = 1 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + \cdots,$$

substituting above expressions into the bilinear equations (7) and collecting the alike power terms, we have

$$\epsilon^1: \quad (D_t + D_x^2)(f\cdot 1) = 0, \quad (SD_t + SD_x^2)(f\cdot 1) = 0, \tag{8}$$

and for $i > 1$

$$\epsilon^i: \quad (D_t + D_x^2)(f\cdot g_i) = 0, \quad (SD_t + SD_x^2)(f\cdot g_i) = 0. \tag{9}$$

From the equations (8), we have

$$f_t + f_{xx} = 0, \ \ \mathcal{D}(f_t + f_{xx}) = 0.$$

Thus we may take

$$f = \mathrm{e}^{kx - k^2 t + \theta\xi}$$

where k is an ordinary constant and ξ is a Grassmann odd constant. On the other hand, with this f, our equations (9) yield

$$g_{i,t} - g_{i,xx} + 2kg_{i,x} = 0,$$

and

$$f(\mathcal{D} - \xi - \theta k)(g_{i,t} - g_{i,xx} + 2kg_{i,x}) = 0.$$

Therefore we can choose

$$g_i = \mathrm{e}^{k_i x + k_i(k_i - 2k)t + \theta\xi_i}$$

as our solutions. Finally, we have

$$\tau_1 = \mathrm{e}^{kx - k^2 t + \theta\xi}, \ \ \tau_2 = 1 + \sum_{i=1}^{N} \mathrm{e}^{k_i x + k_i(k_i - 2k)t + \theta\xi_i}.$$

We see that this type of solutions is characterized by the fact that the coupling constants are zero. Therefore it is the analogy of the fission and fusion soliton of the BK system [17][23](see also [8][7]).

4 Bäcklund Transformation

Bäcklund transformation is an important ingredient of soliton theory. It can be useful for solution construction and serves as a characteristic of integrability for a given system. In this section, we will derive a bilinear BT for our sTB system. We follow Nakamura and Hirota [21] and our results are summarized in the following

Proposition 1 *Suppose that (τ_1, τ_2) is a solution of the equation (7), and $(\bar{\tau}_1, \bar{\tau}_2)$ satisfies the following relations*

$$D_x(\bar{\tau}_1 \cdot \tau_2 + \tau_1 \cdot \bar{\tau}_2) = 0, \qquad S(\bar{\tau}_1 \cdot \tau_2 + \tau_1 \cdot \bar{\tau}_2) = 0, \tag{10}$$

$$D_t(\tau_1 \cdot \bar{\tau}_2) - D_x^2(\bar{\tau}_1 \cdot \tau_2) = 0, \ \ D_t(\bar{\tau}_1 \cdot \tau_2) - D_x^2(\tau_1 \cdot \bar{\tau}_2) = 0, \tag{11}$$

$$SD_t\tau_1 \cdot \bar{\tau}_2 + \frac{1}{2}SD_x^2\tau_1 \cdot \bar{\tau}_2 - \frac{1}{2}SD_x^2\bar{\tau}_1 \cdot \tau_2 = 0, \tag{12}$$

$$SD_t\bar{\tau}_1 \cdot \tau_2 + \frac{1}{2}SD_x^2\bar{\tau}_1 \cdot \tau_2 - \frac{1}{2}SD_x^2\tau_1 \cdot \bar{\tau}_2 = 0, \tag{13}$$

is the another solution of (7).

Proof: We consider the following

$$\begin{aligned}
\mathbb{P}_1 &= D_x\Big[[(D_t+D_x^2)\tau_1\cdot\tau_2]\cdot\bar{\tau}_1\bar{\tau}_2-\tau_1\tau_2\cdot[(D_t+D_x^2)\bar{\tau}_1\cdot\bar{\tau}_2]\Big],\\
\mathbb{P}_2 &= \Big[[S(D_t+D_x^2)\tau_1\cdot\tau_2]\bar{\tau}_1\bar{\tau}_2+\tau_1\tau_2[S(D_t+D_x^2)\bar{\tau}_1\cdot\bar{\tau}_2]\Big].
\end{aligned}$$

We will show that above equations (10-13) imply $\mathbb{P}_1=0$ and $\mathbb{P}_2=0$. The case of $\mathbb{P}_1$ can be verified as in [21], so we will concentrate on $\mathbb{P}_2$. We will use various bilinear identities

which are presented in the Appendix.

$$
\begin{aligned}
\mathbb{P}_2 \quad & = \quad (SD_t\tau_1\cdot\tau_2)\bar\tau_1\bar\tau_2+\tau_1\tau_2(SD_t\bar\tau_1\cdot\bar\tau_2)+(SD_x^2\tau_1\cdot\tau_2)\bar\tau_1\bar\tau_2+\tau_1\tau_2(SD_x^2\bar\tau_1\cdot\bar\tau_2) \\
& \overset{(A.1),(A.3)}{=} \quad (SD_t\tau_1\cdot\bar\tau_2)\bar\tau_1\tau_2+\tau_1\bar\tau_2(SD_t\bar\tau_1\cdot\tau_2)+(S\tau_1\cdot\bar\tau_1)(D_t\bar\tau_2\cdot\tau_2)+(D_t\tau_1\cdot\bar\tau_1)(S\bar\tau_2\cdot\tau_2) \\
& \quad +D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2+\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\}-(S\tau_1\cdot\tau_2)(D_x^2\bar\tau_1\cdot\bar\tau_2) \\
& \quad -(S\bar\tau_1\cdot\bar\tau_2)(D_x^2\tau_1\cdot\tau_2)+2(SD_x\tau_1\cdot\tau_2)(D_x\bar\tau_1\cdot\bar\tau_2)+2(SD_x\bar\tau_1\cdot\bar\tau_2)(D_x\tau_1\cdot\tau_2) \\
& \overset{(A.2)}{=} \quad (SD_t\tau_1\cdot\bar\tau_2)\bar\tau_1\tau_2+\tau_1\bar\tau_2(SD_t\bar\tau_1\cdot\tau_2)+(S\tau_1\cdot\tau_2)(D_t\bar\tau_2\cdot\bar\tau_1)+(D_t\tau_1\cdot\tau_2)(S\bar\tau_2\cdot\bar\tau_1) \\
& \quad +(S\tau_1\cdot\bar\tau_2)(D_t\bar\tau_1\cdot\tau_2)+(D_t\tau_1\cdot\bar\tau_2)(S\bar\tau_1\cdot\tau_2)+D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2 \\
& \quad +\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\}-(S\tau_1\cdot\tau_2)(D_x^2\bar\tau_2\cdot\bar\tau_1)-(D_x^2\tau_1\cdot\tau_2)(S\bar\tau_1\cdot\bar\tau_2) \\
& \quad +2(SD_x\tau_1\cdot\tau_2)(D_x\bar\tau_1\cdot\bar\tau_2)+2(SD_x\bar\tau_1\cdot\bar\tau_2)(D_x\tau_1\cdot\tau_2) \\
& = \quad (SD_t\tau_1\cdot\bar\tau_2)\bar\tau_1\tau_2+\tau_1\bar\tau_2(SD_t\bar\tau_1\cdot\tau_2)+(S\tau_1\cdot\bar\tau_2)(D_t\bar\tau_1\cdot\tau_2)+(D_t\tau_1\cdot\bar\tau_2)(S\bar\tau_1\cdot\tau_2) \\
& \quad +D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2+\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\}+2(SD_x\tau_1\cdot\tau_2)(D_x\bar\tau_1\cdot\bar\tau_2) \\
& \quad +2(SD_x\bar\tau_1\cdot\bar\tau_2)(D_x\tau_1\cdot\tau_2) \\
& \overset{(A.4)}{=} \quad \left[(SD_t+\frac{1}{2}SD_x^2)\tau_1\cdot\bar\tau_2\right]\bar\tau_1\tau_2+\tau_1\bar\tau_2\left[(SD_t+\frac{1}{2}SD_x^2)\bar\tau_1\cdot\tau_2\right] \\
& \quad +(S\tau_1\cdot\bar\tau_2)\left[(D_t+\frac{1}{2}D_x^2)\bar\tau_1\cdot\tau_2\right]+\left[(D_t+\frac{1}{2}D_x^2)\tau_1\cdot\bar\tau_2\right](S\bar\tau_1\cdot\tau_2) \\
& \quad +\frac{1}{2}D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2+\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\}-(SD_x\bar\tau_2\cdot\tau_1)(D_x\tau_2\cdot\bar\tau_1) \\
& \quad -(SD_x\tau_2\cdot\bar\tau_1)(D_x\bar\tau_2\cdot\tau_1) \\
& \overset{(11-13)}{=} \quad \frac{1}{2}(SD_x^2\bar\tau_1\cdot\tau_2)\bar\tau_1\tau_2+\frac{1}{2}\tau_1\bar\tau_2(SD_x^2\tau_1\cdot\bar\tau_2)+(S\tau_1\cdot\bar\tau_2)\left[D_x^2\tau_1\cdot\bar\tau_2+\frac{1}{2}D_x^2\bar\tau_1\cdot\tau_2\right] \\
& \quad +(S\bar\tau_1\cdot\tau_2)\left[D_x^2\bar\tau_1\cdot\tau_2+\frac{1}{2}D_x^2\tau_1\cdot\bar\tau_2\right]+\frac{1}{2}D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2+\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\} \\
& \quad -(SD_x\bar\tau_2\cdot\tau_1)(D_x\tau_2\cdot\bar\tau_1)-(SD_x\tau_2\cdot\bar\tau_1)(D_x\bar\tau_2\cdot\tau_1) \\
& = \quad \frac{1}{2}(SD_x^2\bar\tau_1\cdot\tau_2)\bar\tau_1\tau_2+\frac{1}{2}\tau_1\bar\tau_2(SD_x^2\tau_1\cdot\bar\tau_2)+\left[(S\tau_1\cdot\bar\tau_2)+\frac{1}{2}(S\bar\tau_1\cdot\tau_2)\right](D_x^2\tau_1\cdot\bar\tau_2) \\
& \quad +\left[(S\bar\tau_1\cdot\tau_2)+\frac{1}{2}(S\tau_1\cdot\bar\tau_2)\right](D_x^2\bar\tau_1\cdot\tau_2)+\frac{1}{2}D_x^2\{(S\tau_1\cdot\bar\tau_2)\cdot\bar\tau_1\tau_2+\tau_1\bar\tau_2\cdot(S\bar\tau_1\cdot\tau_2)\} \\
& \quad +(SD_x\tau_1\cdot\bar\tau_2)(D_x\bar\tau_1\cdot\tau_2)+(SD_x\bar\tau_1\cdot\tau_2)(D_x\tau_1\cdot\bar\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
&\stackrel{(10)}{=} \frac{1}{2}\Big[(SD_x^2\bar{\tau}_1\cdot\tau_2)\bar{\tau}_1\tau_2+(S\bar{\tau}_1\cdot\tau_2)(D_x^2\bar{\tau}_1\cdot\tau_2)-D_x^2[(S\bar{\tau}_1\cdot\tau_2)\cdot\bar{\tau}_1\tau_2]\\
&\quad -2(SD_x\bar{\tau}_1\cdot\tau_2)(D_x\bar{\tau}_1\cdot\tau_2)\Big]+\frac{1}{2}\Big[(SD_x^2\tau_1\cdot\bar{\tau}_2)\tau_1\bar{\tau}_2+(S\tau_1\cdot\bar{\tau}_2)(D_x^2\tau_1\cdot\bar{\tau}_2)\\
&\quad -D_x^2[(S\tau_1\cdot\bar{\tau}_2)\cdot\tau_1\bar{\tau}_2]-2(SD_x\tau_1\cdot\bar{\tau}_2)(D_x\tau_1\cdot\bar{\tau}_2)\Big]\\
&\stackrel{(A.5)}{=} 0,
\end{aligned}
$$

thus our proposition is proved.

5 Discussion

We constructed the bilinear form for the sTB system, then we presented a class of solutions. This kind of solutions is the generalization of the fusion and fission solutions of the cB equation [17][23] (see [24] for other systems which possess this sort of solutions). Apart from the fusion and fission solitons, there is another type of solitons, which was constructed by Kaup [10] in the framework of inverse scattering transformation and by Hirota and Satsuma using bilinear formulism [9]. However, the present version of bilinearization does not apparently allow one to calculate this sort of solution in the supersymmetric case. It would be interesting to find out if such soliton solutions exist or not in the supersymmetric case.

Acknowledgements We should like to thank Xing-Biao Hu for interesting discussions. It is pleasure to thank the anonymous referees for pointing out many typos of early version of the paper. This work is supported in part by NNSFC under the grant number 10231050 and the Ministry of Education of China.

Appendix: Some Bilinear Identities

In this Appendix, we list the relevant bilinear identities, which can be proved directly. Here a, b, c and d are arbitrary even functions of the independent variable x, t and θ.

$$
\begin{aligned}
(SD_ta\cdot b)cd+ab(SD_tc\cdot d) &= (SD_ta\cdot d)cb+ad(SD_tc\cdot b)\\
&\quad +(Sa\cdot c)(D_td\cdot b)+(Sd\cdot b)(D_ta\cdot c),
\end{aligned}\tag{A.1}
$$

$$
\begin{aligned}
(Sa\cdot b)(D_tc\cdot d)+(D_ta\cdot b)(Sc\cdot d) &= (Sa\cdot d)(D_tc\cdot b)+(D_ta\cdot d)(Sc\cdot b)\\
&\quad +(Sa\cdot c)(D_tb\cdot d)+(D_ta\cdot c)(Sb\cdot d),
\end{aligned}\tag{A.2}
$$

$$
\begin{aligned}
(SD_x^2a\cdot b)cd+ab(SD_x^2c\cdot d) &= D_x^2\{(Sa\cdot d)\cdot cb+ad\cdot(Sc\cdot b)\}\\
&\quad -(Sa\cdot b)D_x^2(c\cdot d)-(Sc\cdot d)D_x^2(a\cdot b)\\
&\quad +2(SD_xa\cdot b)D(c\cdot d)+2(SD_xc\cdot d)(D_xa\cdot b),
\end{aligned}\tag{A.3}
$$

$$
\begin{aligned}
(SD_x^2 a\cdot b)cd + ab(SD_x^2 c\cdot d) \quad = \quad & D_x^2\{(Sa\cdot b)\cdot cd + ab\cdot(Sc\cdot d)\} \\
& -(Sa\cdot b)(D_x^2 c\cdot d) - (Sc\cdot d)(D_x^2 a\cdot b) \\
& -4(SD_x b\cdot c)(D_x d\cdot a) - 4(SD_x d\cdot a)(D_x b\cdot c) \\
& -2(SD_x a\cdot b)(D_x c\cdot d) - 2(SD_x c\cdot d)(D_x a\cdot b), \quad \text{(A.4)}
\end{aligned}
$$

$$
D_x^2[(Sa\cdot b)\cdot ab] - (SD_x^2 a\cdot b)ab - (Sa\cdot b)D_x^2(a\cdot b) + 2(SD_x a\cdot b)(D_x a\cdot b) = 0. \quad \text{(A.5)}
$$

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